Parallel-Coupled Transmission-Line-Resonator Filters*

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Summary—This paper describes the synthesis of band-pass transmission-line filters consisting of series of half-wavelength resonant conductors such as strips. The design differs from the usual end-coupled strip configuration in that successive strips are parallel coupled along a distance of a quarter-wavelength. The resulting coupling between resonators is partly electric and partly magnetic. Several important advantages are gained by this arrangement: 1) the length of the filter is approximately half that of the end-coupled type; 2) the gaps are larger and therefore less critical; and 3) the insertion-loss curve is symmetrical on a frequency scale with the first spurious response occurring at three times the center frequency of the pass band.

Formulas are derived for the parallel-coupled-resonator transmission-line filter that permit accurate design for Tchebycheff, maximally flat, or any other physically realizable response. The formulas are theoretically exact in the limit of zero bandwidth, but frequency-response calculations show them to give good results for bandwidths up to about 30 per cent. An experimental strip-line filter of this type has been constructed, and the data given in this paper show that excellent performance has been obtained.

INTRODUCTION

As shown in Fig. 1(a),[1–3] multiple-coupled-resonator band-pass filters in strip line (or other TEM transmission line) have been most commonly composed of half-wavelength strips coupled end to end. In this paper, an alternative arrangement is treated in which the half-wavelength strips are parallel-coupled, as in Fig. 1(b) and 1(c). Parallel coupling offers a number of important advantages over end coupling: 1) the length of the filter is reduced approximately by half; 2) a symmetrical insertion-loss-vs-frequency response is obtained with the first spurious response occurring at three times the center frequency, and 3) a much larger gap between adjacent strips is permitted. The last advantage is of particular importance, since it eases the tolerance on the gaps for a given bandwidth, or permits a broader bandwidth for a given tolerance. Furthermore, the larger gap permits a higher power rating of the filter.

Formulas have been derived that allow the parallel-coupled-resonator filter to be designed in a straightforward manner, and to have any desired physically realizable response, such as maximally flat or equal-ripple. The accuracy of the design formulas has been checked by exact computation, and they have been found to give excellent results for bandwidths up to about 20 per cent in the case of maximally flat response and 30 per cent in the case of equal ripple response. The method of analysis is basically that used in an earlier paper[3] for a number of other types of direct-coupled-resonator filters. In this method, the lumped-constant low-pass prototype filter having the desired response is made equivalent to a set of either series- or parallel-resonant LC arms interconnected by broad-band quarter-wavelength transformers. An approximate equivalence then is established between the latter circuit and the actual coupled-resonator structure.

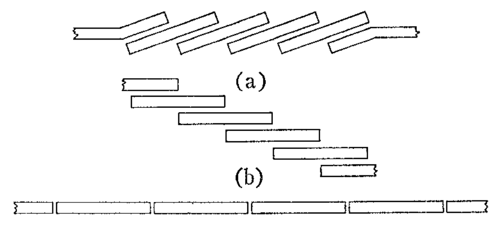

Fig. 1—Coupled-resonator strip-line filters: (a) end coupled; (b) and (c) parallel coupled.

DESIGN PROCEDURE

The design formulas for an n-resonator parallel-coupled filter are given in Table I. They utilize the element values $g_1, g_2, \cdots, g_n$ of the prototype low-pass filter, which may be computed for either maximally flat or equal-ripple response by means of formulas given in Table II.[3] The schematic diagram in Fig. 2 shows that the filter is assembled from $n+1$ sections, resulting in a structure containing n resonators. The sections are of equal length (one-quarter wavelength at the center frequency), and their electrical design is completely specified by two characteristic impedances—Z_{oe} of the even-mode wave on the parallel conductors, and Z_{oo} of the odd-mode wave. These characteristic impedances have previously been defined, and formulas and graphs relating them to the dimensions of the cross section are available.[4] The total filter structure always will be symmetrical for maximally flat or equal-ripple response.

In the design of a parallel-coupled-resonator filter, one should first select the type of response function and the number of resonators that will yield the desired in-

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[1] E. G. Fubini, W. E. Fromm, and H. S. Keen, "Microwave applications of high-Q strip components," IRE CONVENTION RECORD, Pt. 8, pp. 98–103; March, 1954.

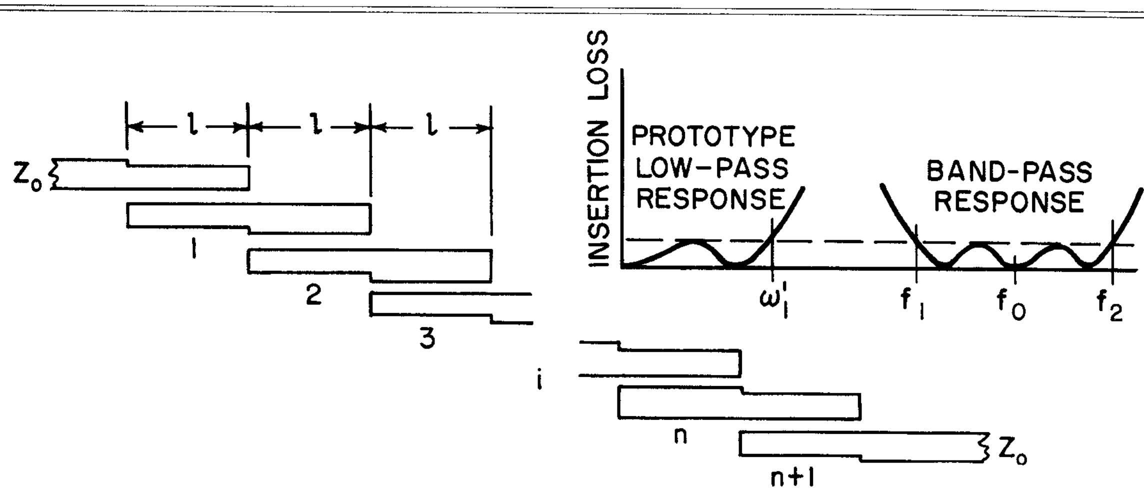
[2] E. A. Bradley, "Design and development of strip-line filters," IRE TRANS., vol. MTT-4, pp. 86–93; April, 1956.

[3] S. B. Cohn, "Direct-coupled-resonator filters," PROC. IRE, vol. 45, pp. 187–196; February, 1957.

[4] S. B. Cohn, "Shielded coupled-strip transmission line," IRE TRANS., vol. MTT-3, pp. 29–38, October, 1955.

TABLE I

FORMULAS FOR PARALLEL-COUPLED TRANSMISSION-LINE-RESONATOR FILTER



(For either maximally-flat or equal-ripple response, the structure is symmetrical.)

$l = \lambda_0/4$, where λ_0 = wavelength in transmission line at f_0

$f_0 = (f_1 + f_2)/2$

$$Z_{oe_i} = Z_o \left\{ 1 + \frac{Z_o}{K_{i-1,i}} + \left(\frac{Z_o}{K_{i-1,i}} \right)^2 \right\}, \qquad i = 1 \text{ to } n+1$$

$$Z_{oo_i} = Z_o \left\{ 1 - \frac{Z_o}{K_{i-1,i}} + \left(\frac{Z_o}{K_{i-1,i}} \right)^2 \right\}, \qquad i = 1 \text{ to } n+1$$

$$\frac{Z_o}{K_{i-1,i}} = \frac{\pi}{\omega_1'} \left(\frac{f_2 - f_1}{f_2 + f_1} \right) \left(\frac{1}{g_{i-1} g_i} \right)^{1/2}$$

$g_1, g_2, \cdots, g_n$ = prototype elements in farads and henries from Table II.

$$g_0 = \frac{\pi}{\omega_1'} \left(\frac{f_2 - f_1}{f_2 + f_1} \right)$$

$$g_{n+1} = \frac{\pi}{r\omega_1'} \left(\frac{f_2 - f_1}{f_2 + f_1} \right)$$

r = right-hand load resistance in schematic of Table II.
ω_1' = pass band edge of prototype filter.
f_1, f_2 = corresponding pass band edges of transmission-line filter.
Z_{oe_i} = even-mode characteristic impedance with respect to ground of each conductor in ith section.
Z_{oo_i} = odd-mode characteristic impedance with respect to ground of each conductor in ith section.

(The dimensions for strip-line construction may be obtained as function of Z_{oe_i} and Z_{oo_i} from Cohn.[4])

sertion-loss function in the pass and stop bands. This may be done with the aid of the insertion-loss formulas in Table II and the following approximate relationship between the frequency scales of the prototype filter and the parallel-coupled-resonator filter:

$$\omega' = 2\omega_1' \left(\frac{f - f_0}{f_2 - f_1} \right). \qquad (1)$$

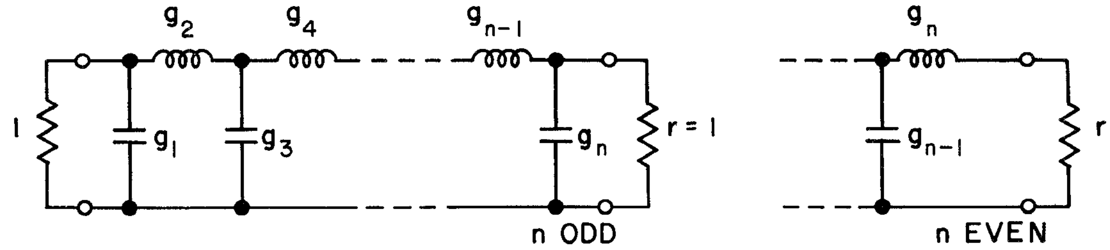
The exact response curves and the bandwidth-error curve (which appear later) show how response errors are introduced when the relative bandwidth exceeds 0.1. The bandwidth error may be reduced by preadjusting the design with the aid of these curves.

Next, the element values $g_1, g_2, \cdots, g_n$ may be computed, and in terms of these the Z_{oe} and Z_{oo} values of each of the $n+1$ sections. Then, the transmission-line dimensions in each section should be designed to yield these characteristic impedances. This may be done easily in the case of thin strip conductors, and with somewhat more difficulty in the case of thick strips, by means of the available graphs and formulas.[4] It is noted that the strip widths and spacings will, in general, differ from section to section, and hence the width of the resonators will not be constant, as shown in Fig. 2. This variation in width is necessary to compensate for differences in the coupling gap. The amount of variation decreases as the bandwidth is reduced and, except perhaps for the first and last section, the variation is negligible in the case of bandwidths less than a few per cent.

One further important step in the filter design is to alter the length of the resonators to compensate for fringing capacitance at their ends. This may be done

TABLE II

PROTOTYPE LOW-PASS FILTER AND ITS DESIGN EQUATIONS FOR MAXIMALLY-FLAT AND TCHEBYCHEFF RESPONSE

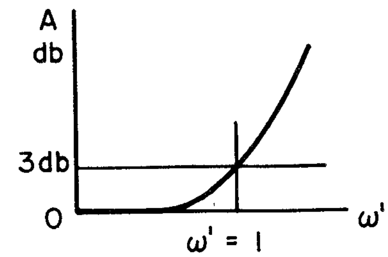


Maximally-Flat Response

$r = 1$ for all n

$$g_k = 2 \sin\left[\frac{(2k-1)\pi}{2n}\right], \qquad k = 1, 2, \cdots, n$$

$$A = 10 \log_{10} (1 + \omega'^{2n})\text{db}$$



Tchebycheff Response

$r = 1$ for n odd, $\quad r = \tanh^2 (\beta/4)$ for n even

$$g_1 = 2a_1/\gamma$$

$$g_k = \frac{4a_{k-1}a_k}{b_{k-1}g_{k-1}}, \qquad k = 2, 3, \cdots, n$$

check: $g_n = g_1 r$

$$a_k = \sin\left[\frac{(2k-1)\pi}{2n}\right], \qquad k = 1, 2, \cdots, n$$

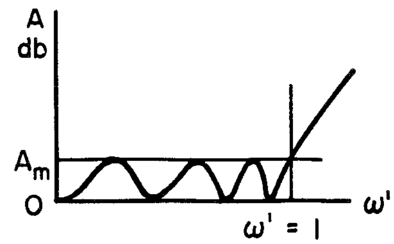
$$b_k = \gamma^2 + \sin^2\left(\frac{k\pi}{n}\right), \qquad k = 1, 2, \cdots, n$$

$$\beta = \log_e\left(\coth \frac{A_m}{17.37}\right), \qquad A_m \text{ in db}$$

$$\gamma = \sinh\left(\frac{\beta}{2n}\right)$$

$$A = 10 \log_{10} [1 + (10^{A_m/10} - 1) \cos^2 (n \cos^{-1} \omega')]\text{db}, \qquad \omega' \leq 1$$

$$A = 10 \log_{10} [1 + (10^{A_m/10} - 1) \cosh^2 (n \cosh^{-1} \omega')]\text{db}, \qquad \omega' \geq 1$$



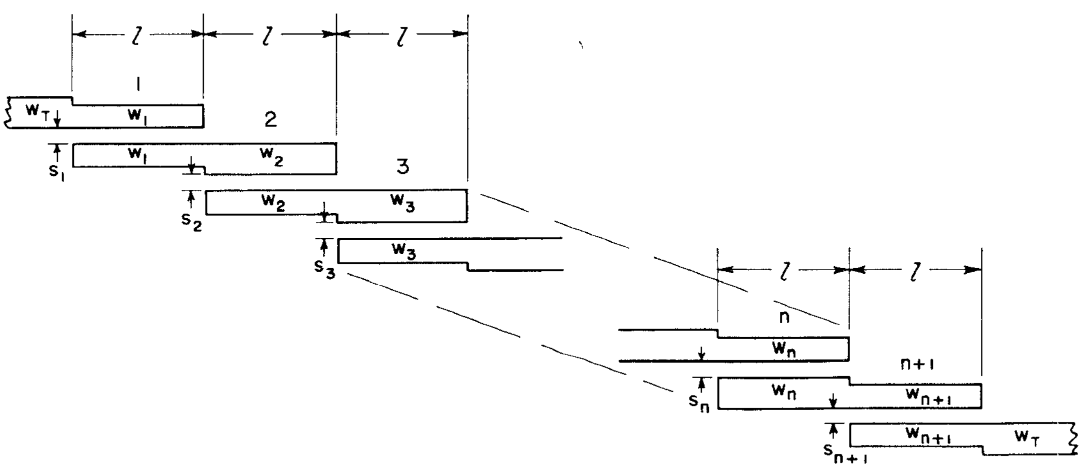


Fig. 2—Basic dimensions of the strip-line parallel-coupled-resonator filter.

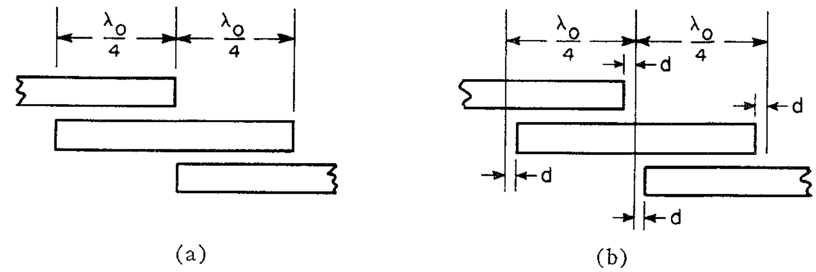


Fig. 3—(a) Strip-resonator design neglecting fringing capacitance at ends. (b) Suggested compensation for fringing capacitance at ends of resonators.

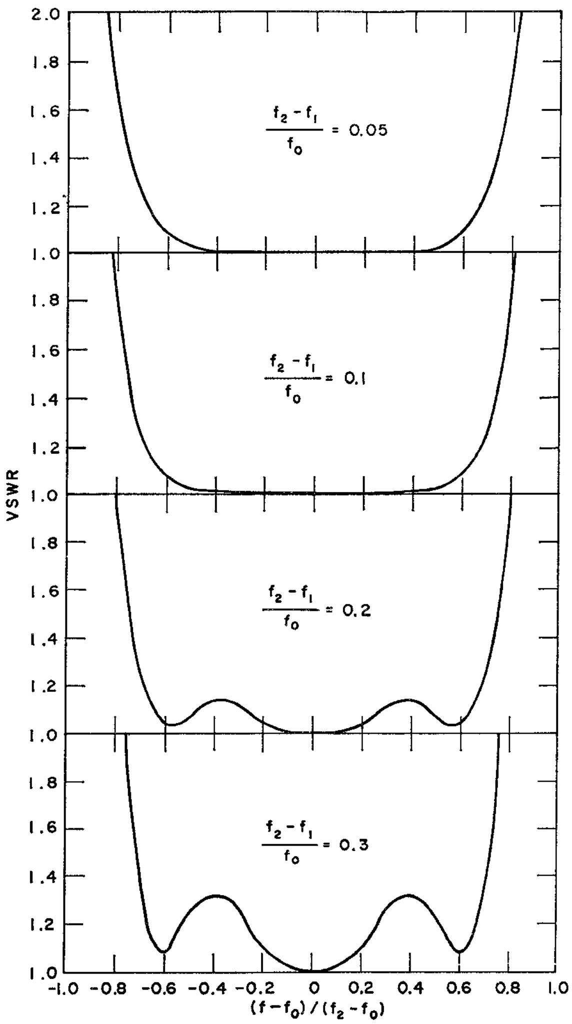


Fig. 4—VSWR curves of six-resonator, parallel-coupled filter designed for maximally-flat response.

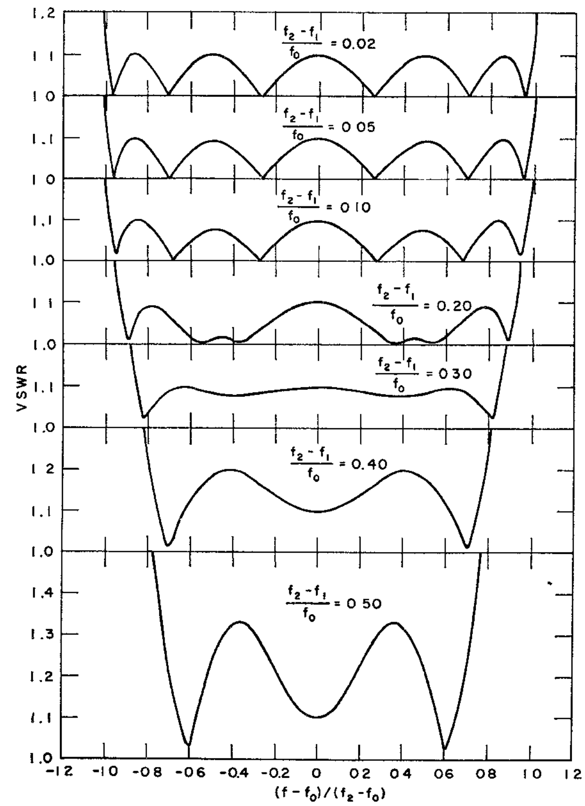


Fig. 5—VSWR curves of six-resonator, parallel-coupled filter designed for equal-ripple response—vswr = 1.10.

as shown in Fig. 3(b), where each end is cut back by the average length, d. The dimension d would be expected to be somewhat less than $(C_f'/\epsilon)\cdot(b/2)$, where C_f'/ϵ is a quantity plotted elsewhere as a function of the strip-thickness-to-plate-spacing ratio, t/b.[5] Measurements on the experimental filter described below indicate that d should be approximately 0.75 $(C_f'/\epsilon)\cdot(b/2)$. For very thin strips, this would be 0.75 $(0.220b) = 0.165b$. In wide-bandwidth filters an error in d would be unimportant, while in narrow-bandwidth filters the error would be cancelled by the tuning adjustments that are already needed to overcome the effects of constructional tolerances. This tuning may be done, for example, by means of screws or dielectric slugs in high-electric-field regions of the resonators.

[5] S. B. Cohn, "Problems in strip transmission line," IRE Trans., Vol. MTT-3, pp. 119–126; March, 1955.

Verification of Design Accuracy

Because of the approximations necessary in the derivation of the design formulas, their accuracy was checked theoretically. This was done by exact computation of the insertion-loss and vswr responses of the actual transmission-line filter networks specified by the formulas for various bandwidths and for either maximally flat or Tchebycheff response. The computations were performed on an electronic computer by a matrix-

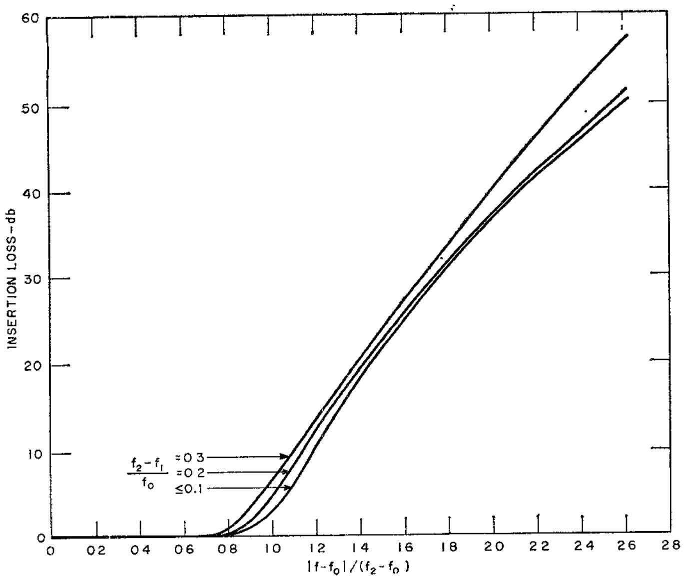


Fig. 6—Insertion-loss curves of six-resonator, parallel-coupled filter designed for maximally-flat response.

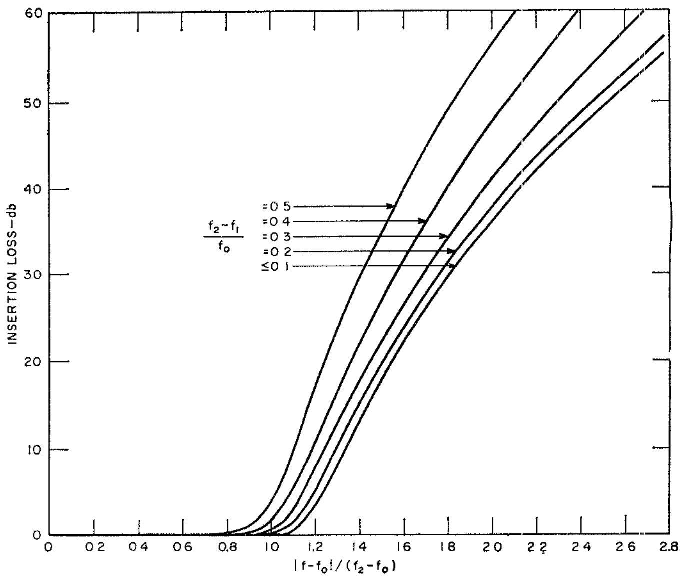


Fig. 7—Insertion-loss curves of six-resonator, parallel-coupled filter designed for equal-ripple response—vswr = 1.10.

multiplication method.[6] Because the filter is symmetrical, it is sufficient to compute the matrix of half the filter. The insertion loss and vswr then were computed as functions of the elements of this matrix.

The pass band vswr curves for six-resonator maximally flat filters of various bandwidths are shown in Fig. 4. The relative bandwidth is defined as

$$W = \frac{f_2 - f_1}{f_0}, \tag{2}$$

where $f_0 = (f_1 + f_2)/2$, and f_1 and f_2 are the 3-db points of the response curve. The response is seen to be truly maximally flat for relative bandwidths up to 0.1, while at 0.2 the deviation from maximally flat is slight. At 0.3, however, the deviation would be serious in the more critical applications. The corresponding curves are shown in Fig. 5 for six-resonator filters designed to have an equal ripple vswr of 1.10 in the pass band. In this case, f_1 and f_2 are the pass band limits for the equal ripple-level vswr of 1.10. The desired response is obtained very accurately for relative bandwidths up to 0.05, and gradually deteriorates as the bandwidth is increased further. In spite of this deterioration, the pass band vswr limit of 1.10 is not exceeded for relative bandwidths up to 0.3, and even at 0.4 or 0.5 the response is adequate for many applications. But, it is important to note that these conclusions are drawn from specific cases considered, and may vary somewhat with other numbers of resonators or with other equal ripple levels.

The insertion-loss curves for the above cases are plotted against a normalized frequency scale in Fig. 6 and Fig. 7. Because the response is symmetrical with

[6] P. I. Richards, "Applications of matrix algebra to filter theory," Proc. IRE, vol. 34, pp. 145P–150P; March, 1946.

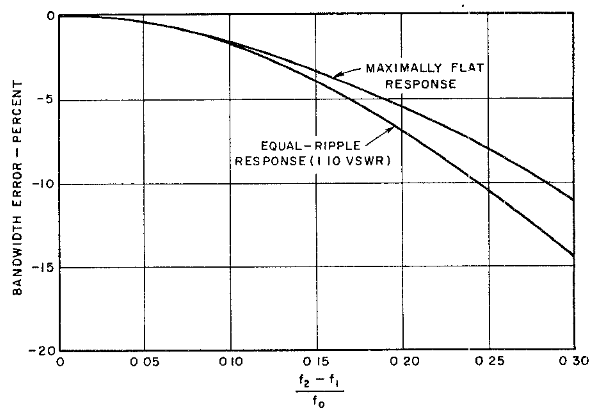


Fig. 8—Bandwidth error vs relative bandwidth for six-resonator parallel-coupled filter. Error is expressed as a percentage of the *bandwidth;* e.g., at $(f_2 - f_1)/f_0 = 0.10$, corrected bandwidth = 0.098.

frequency, only half of each pass band is shown. In both figures, the insertion loss is virtually identical with that of the prototype function for relative bandwidths up to 0.1, and even for greater bandwidths, the deviation is quite small.

In addition to the errors in response-curve shape that develop as the bandwidth is increased, the predicted bandwidth is slightly in error. In Fig. 8 the bandwidth error in per cent is plotted vs relative bandwidth for the cases considered above. It is seen that the discrepancy does not exceed 2 per cent of the bandwidth for relative bandwidths up to 0.1, and is only about 6 per cent at 0.2. The actual bandwidth of the filter appears to be always less than the value assumed in the design, and therefore, in the case of the wider bandwidths, it would be desirable in calculating the parameters of a given filter to use a somewhat larger bandwidth than actually is required. Fig. 8 may be used as an approximate guide in selecting the bandwidth design value.

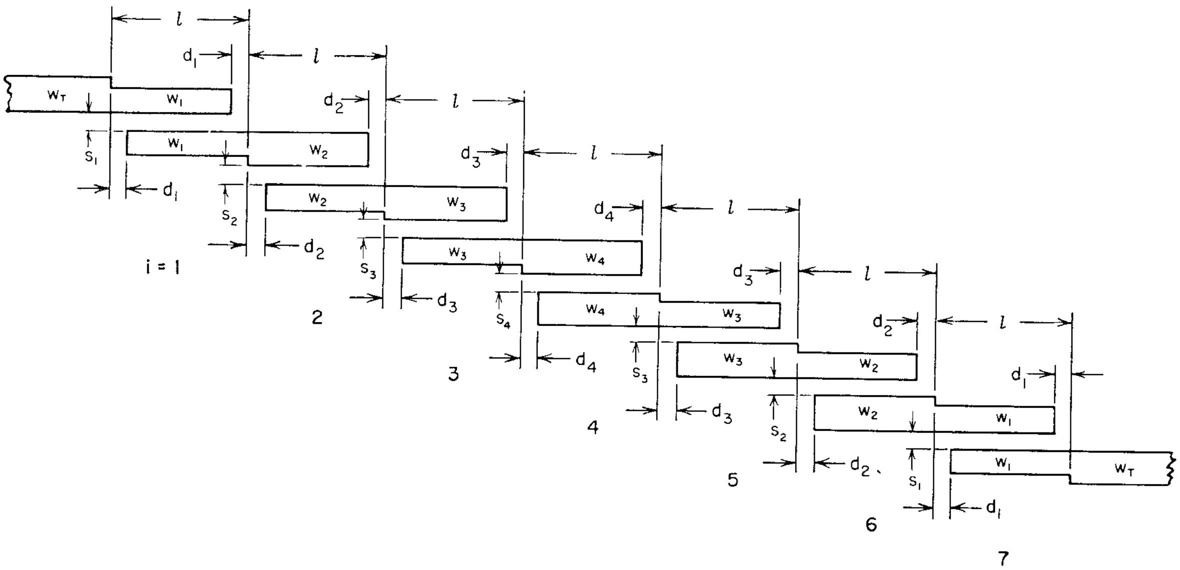


Fig. 9—Layout of parallel-coupled-resonator filter.

TABLE III
DESIGN PARAMETERS FOR EXPERIMENTAL PARALLEL-COUPLED STRIP-LINE-RESONATOR FILTER

i	$Z_o / K_{i-1,i}$	Z_{oei}	Z_{ooi}	w_i	s_i	d_i
1	0.449	82.5 ohms	37.6 ohms	0.236 inch	0.021 inch	0.073 inch
2	0.1529	58.8 ohms	43.5 ohms	0.346 inch	0.110 inch	0.084 inch
3	0.1038	55.7 ohms	45.3 ohms	0.360 inch	0.158 inch	0.085 inch
4	0.0976	55.4 ohms	45.6 ohms	0.361 inch	0.163 inch	0.085 inch

DESIGN OF EXPERIMENTAL FILTER

A strip-line parallel-coupled-resonator filter has been constructed and tested in order to demonstrate the feasibility of this type of design. As shown in Fig. 9, the filter has six resonators. The two ground planes were spaced 0.5 inch apart by a polystyrene dielectric. The strips are of copper foil, 0.0017 inch thick. The filter was designed for 10-per cent bandwidth, centered at 1200 mc, and an equal ripple pass band vswr of 1.10. The exact theoretical vswr and the insertion-loss curves of the actual transmission-line filter network, as calculated by an electronic computer, are shown in Fig. 5 and Fig. 7. In each figure the appropriate curve is labeled $(f_2-f_1)/f_0=0.1$. The details of the design procedure and a discussion of the experimental results are given below.

Fig. 9 shows that the filter has seven sections, but because of the symmetry of the filter, it is necessary to compute the parameters of only the first four sections. The first step in the design procedure is to compute the element values g_1, g_2, g_3, and g_4 of the prototype seven-element, low-pass filter. This is done by means of the formulas for Tchebycheff response in Table II, with $n=7$ and $A_m=0.00986$ db (which corresponds to an input vswr of 1.10). The resulting values are

$$g_1 = 0.77968, \quad g_2 = 1.35921, \quad g_3 = 1.68800, \quad g_4 = 1.53454.$$

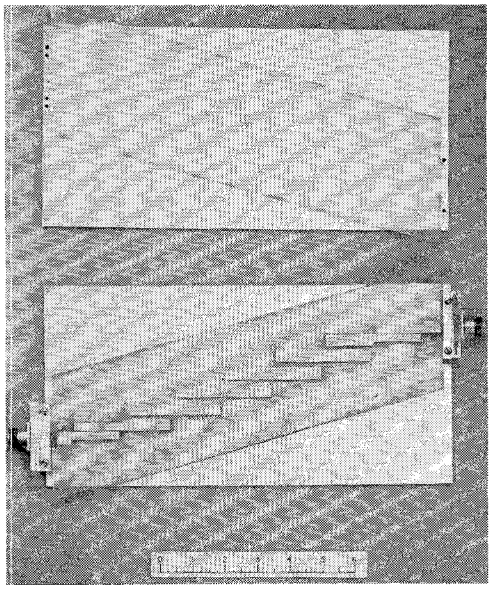
Next, the quantity $Z_o/K_{i-1,i}$ and the even- and odd-mode characteristic impedances Z_{oe_i} and Z_{oo_i} are computed from the formulas in Table I. In this calculation $(f_2-f_1)/f_0$ is set equal to 0.1, and Z_o to 50 ohms. Then the strip widths w_i and separations s_i of the various sections are obtained from the nomograms of a previous paper,[4] with ϵ_r equal to 2.55, b equal to 0.5 inch, and strip thickness assumed to be zero. Finally, the resonators are shortened at their ends by d_i to compensate for fringing capacitance. The various quantities referred to above are contained in Table III.[7]

A further quantity required is the width w_T of the terminating 50-ohm strips. This may be obtained from a graph of characteristic impedance vs strip width.[5,8] The result is $w_T/b=0.744$ or $w_T=0.372$ inch. The section length l is a quarter-wavelength in the dielectric, and hence equals 1.540 inch.

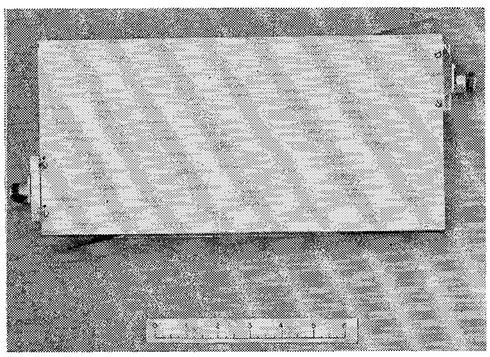
Photographs of the completed filter are shown in Fig. 10. The structure is a sandwich of copper foil separated by a pair of polystyrene plates each machined to have flat surfaces and a thickness of 0.250 inch. Thus, the total ground-plane spacing is one-half inch. The copper foil was cemented to the polystyrene plates with Dow

[7] The values of d_2 in Table III were computed from an approximate formula, and are quite close to the value 0.165b mentioned above. Because the formula is rather complex, and its assumptions are as yet unproved, it is not given here.

[8] S. B. Cohn, "Characteristic impedance of the shielded-strip transmission line," IRE TRANS., vol. MTT-2, pp. 52–57; July, 1954.



(a)



(b)

Fig. 10—Photographic views of parallel-coupled resonator filter: (a) with upper plate removed; (b) completely assembled.

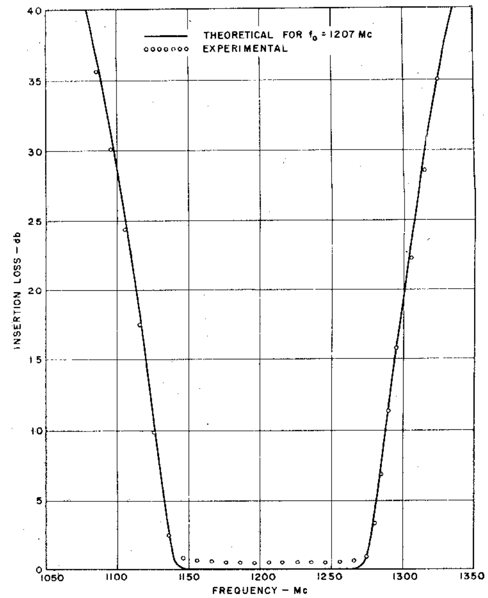


Fig. 11—Insertion loss of parallel-coupled-resonator filter.

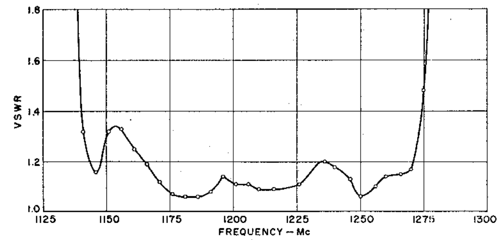


Fig. 12—Measured vswr of parallel-coupled-resonator filter.

Corning XC-271 adhesive, which was applied to the copper and air dried for 45 minutes before pressing the foil onto the polystyrene. The strip circuit then was cut on one surface with a sharp knife and the unwanted foil peeled off.

Data for Experimental Filter

A preliminary model of the parallel-coupled resonator filter was tested first with the resonator ends uncompensated; *i.e.*, with $d_i=0$. The center frequency was lower than the design value by 4.4 per cent. Then, the open-circuited strip ends all were cut back uniformly a distance $d=0.220b$, the value of $(C_f'/\epsilon)/(b/2)$ for very thin strips, and the center frequency was higher than the design value by 1.7 per cent. Finally, the filter was reconstructed with the values of d_i given in Table III. The center frequency of this model is 1207 mc, rather than the design value of 1200 mc—an error of only 0.6 per cent. In the three cases the bandwidth and response curves are quite similar. Although the best pass band response was obtained with the last case, it may have been due to more accurate construction.

Fig. 11 shows a comparison between the measured insertion loss of the filter and the exact theoretical curve for the transmission-line network computed for a center frequency of 1207 mc and neglecting dissipation. Above 2 db, the agreement is excellent. The bandwidth error is seen to be only about 1.0 per cent of the bandwidth. In the pass band, however, disagreement is inevitable because of the finite Q of the strip-line resonators. Nevertheless, the measured pass band insertion loss is quite uniform at 0.5 to 0.7 db.

The input vswr of the filter measured with a 50-ohm termination on the output is shown in Fig. 12. Although the vswr exceeds the theoretical ripple level of 1.10, it would be quite acceptable for most applications. The highest vswr peak in the pass band occurs near the low frequency end, where it reaches a value of 1.33, while the vswr elsewhere in the pass band is under 1.20. It is believed that the vswr could be made to conform more closely to theory if tuning adjustments were provided for the resonators. These adjustments also could serve to set the center frequency of the filter exactly on the desired value.

Derivation of Design Formulas

The analysis of the parallel-coupled-resonator filter is based upon the characteristics of the individual section of Fig. 13(a). The image impedance Z_I and image phase shift β of this section are given in a paper by Jones and

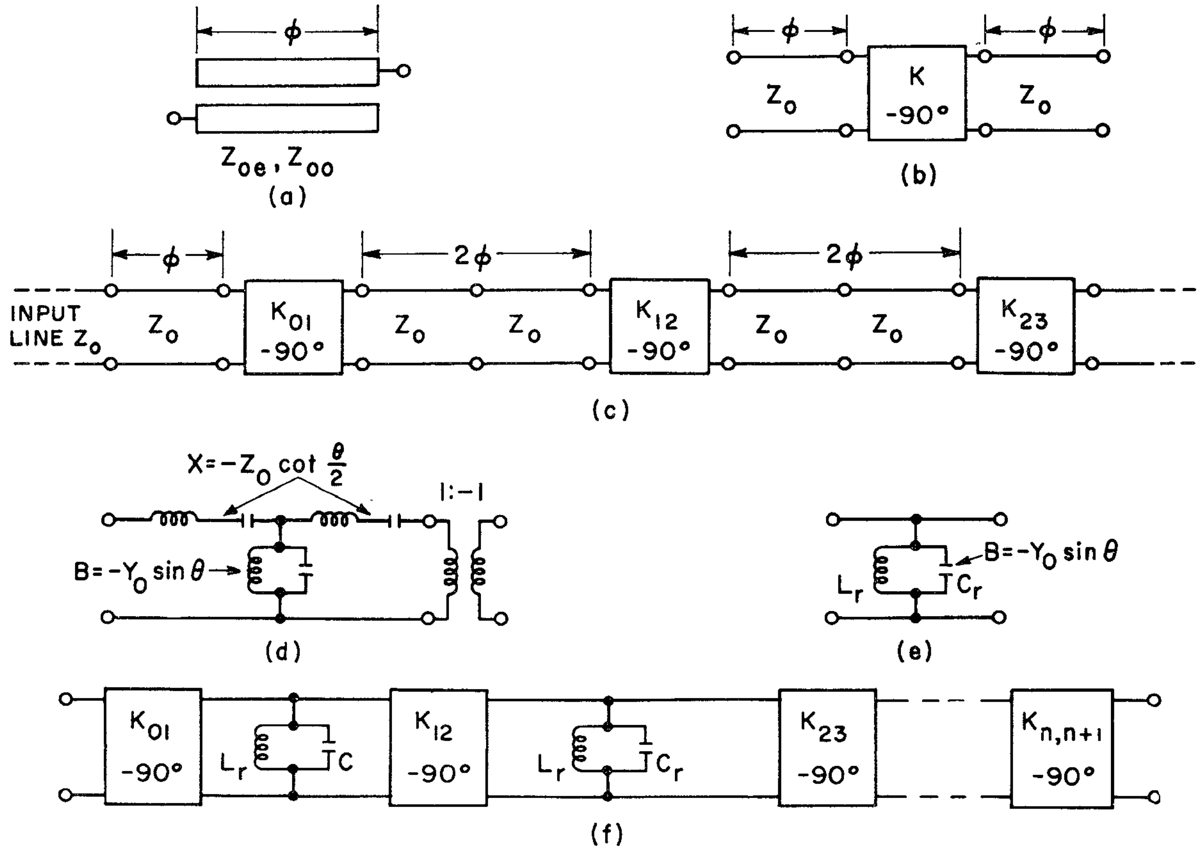


Fig. 13—Development of equivalent circuit of the parallel-coupled-resonator filter.

Bolljahn[9] as follows

$$Z_I = \frac{[(Z_{oe} - Z_{oo})^2 - (Z_{oe} + Z_{oo})^2 \cos^2 \phi]^{1/2}}{2 \sin \phi} \tag{3}$$

$$\cos \beta = \left(\frac{Z_{oe} + Z_{oo}}{Z_{oe} - Z_{oo}}\right) \cos \phi, \tag{4}$$

where ϕ is the electrical length of the coupled transmission lines and Z_{oe} and Z_{oo} are the characteristic impedances of each conductor with respect to ground for the even and odd modes, respectively.

It is now shown that the filter section of Fig. 13(b) is approximately equivalent to that of Fig. 13(a). The box represents an ideal impedance inverter having a constant image impedance, K, and a phase shift of $-90°$ at all frequencies. The image impedance, Z_I, and image phase shift, β, of the section will be derived by means of the $ABCD$ matrix parameters.[5,10]

The $ABCD$ matrix of a transmission line of length ϕ and characteristic impedance Z_o is

$$\begin{bmatrix} \cos \phi & jZ_o \sin \phi \\ \dfrac{j \sin \phi}{Z_o} & \cos \phi \end{bmatrix}.$$

[9] E. M. T. Jones and J. T. Bolljahn, "Coupled-strip-transmission-line filters and directional couplers," IRE Trans., vol. MTT-4, pp. 75–81; April, 1956.

[10] Radio Research Laboratory Staff, "Very High Frequency Techniques," McGraw-Hill Book Co., Inc., New York, N. Y., vol. 2, ch. 26; 1947.

The matrix of the ideal inverter is obtained from this by substituting $\phi = -90°$ and $Z_o = K$, as follows

$$\begin{bmatrix} 0 & -jK \\ \dfrac{-j}{K} & 0 \end{bmatrix}.$$

Therefore, the $ABCD$ matrix of the complete filter section of Fig. 13(b) is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \cos \phi & jZ_o \sin \phi \\ \dfrac{j \sin \phi}{Z_o} & \cos \phi \end{bmatrix} \times \begin{bmatrix} 0 & -jK \\ \dfrac{-j}{K} & 0 \end{bmatrix} \times \begin{bmatrix} \cos \phi & jZ_o \sin \phi \\ \dfrac{j \sin \phi}{Z_o} & \cos \phi \end{bmatrix}$$

$$= \begin{bmatrix} \left(\dfrac{K}{Z_o} + \dfrac{Z_o}{K}\right) \sin \phi \cos \phi & j\left(\dfrac{Z_o^2}{K} \sin^2 \phi - K \cos^2 \phi\right) \\ j\left(\dfrac{K}{Z_o^2} \sin^2 \phi - \dfrac{\cos^2 \phi}{K}\right) & \left(\dfrac{K}{Z_o} + \dfrac{Z_o}{K}\right) \sin \phi \cos \phi \end{bmatrix}.$$

The image parameters are related to the matrix elements by $Z_I = \sqrt{B/C}$ and $\cos \beta = A$, and, hence,

$$Z_I = Z_o \left\{ \frac{\frac{Z_o}{K}\sin^2\phi - \frac{K}{Z_o}\cos^2\phi}{\frac{K}{Z_o}\sin^2\phi - \frac{Z_o}{K}\cos^2\phi} \right\}^{1/2} = Z_o \left\{ \frac{\frac{Z_o}{K} - \left(\frac{K}{Z_o} + \frac{Z_o}{K}\right)\cos^2\phi}{\left(\frac{K}{Z_o} + \frac{Z_o}{K}\right)\sin^2\phi - \frac{Z_o}{K}} \right\}^{1/2}$$

$$= \frac{1}{2\sin\phi} \left\{ \frac{\frac{4Z_o^3}{K}}{\frac{K}{Z_o} + \frac{Z_o}{K} - \frac{Z_o}{K\sin^2\phi}} - \frac{4Z_o^2\left(\frac{K}{Z_o} + \frac{Z_o}{K}\right)\cos^2\phi}{\frac{K}{Z_o} + \frac{Z_o}{K} - \frac{Z_o}{K\sin^2\phi}} \right\}^{1/2} \qquad (5)$$

$$\cos\beta = \left(\frac{K}{Z_o} + \frac{Z_o}{K}\right)\sin\phi\cos\phi. \qquad (6)$$

A comparison between (3) and (5), and between (4) and (6), shows that the sections of Fig. 13(a) and 13(b) would be electrically equivalent if the following conditions were met,

$$\left(\frac{K}{Z_o} + \frac{Z_o}{K}\right)\sin\phi = \frac{Z_{oe} + Z_{oo}}{Z_{oe} - Z_{oo}} \qquad (7)$$

$$\frac{\frac{4Z_o^3}{K}}{\frac{K}{Z_o} + \frac{Z_o}{K} - \frac{Z_o}{K\sin^2\phi}} = (Z_{oe} - Z_{oo})^2 \qquad (8)$$

$$\frac{4Z_o^2\left(\frac{K}{Z_0} + \frac{Z_o}{K}\right)}{\frac{K}{Z_o} + \frac{Z_o}{K} - \frac{Z_o}{K\sin^2\phi}} = (Z_{oe} + Z_{oo})^2. \qquad (9)$$

The presence of $\sin\phi$ in the left side of each of the above makes a frequency-independent equality impossible. However, $\sin\phi$ is stationary in the vicinity of $\phi = 90°$, and hence may be replaced by unity with negligible error over a moderate bandwidth. Thus,

$$\frac{K}{Z_o} + \frac{Z_o}{K} = \frac{Z_{oe} + Z_{oo}}{Z_{oe} - Z_{oo}} \qquad (10)$$

$$\frac{2Z_o^2}{K} = Z_{oe} - Z_{oo} \qquad (11)$$

$$\frac{4Z_o^3}{K}\left(\frac{K}{Z_o} + \frac{Z_o}{K}\right) = (Z_{oe} + Z_{oo})^2. \qquad (12)$$

From these equations, formulas for Z_{oe}/Z_o and Z_{oo}/Z_o as functions of Z_o/K may be found. Only two of the three equations are needed for this purpose, and since the three do not form a consistent set, a judicious choice must be made among them. Examination of (3) to (6) indicates that the third relation is the least important of the three. Therefore, (10) and (11) are solved simultaneously to yield

$$\frac{Z_{oe}}{Z_o} = 1 + \frac{Z_o}{K} + \frac{Z_o^2}{K^2} \qquad (13)$$

$$\frac{Z_{oo}}{Z_o} = 1 - \frac{Z_o}{K} + \frac{Z_o^2}{K^2}. \qquad (14)$$

It may now be noted that, for narrow bandwidth, $Z_o/K \ll 1$ and, therefore, (12) is satisfied approximately by (13) and (14). Thus the effect of the approximations involved in relating the sections of Fig. 13(a) and 13(b) is negligible for narrow bandwidth, but increases in importance as the bandwidth is increased.

When the individual sections of the filter are assembled, it is seen that the original parallel-coupled-resonator filter of Fig. 1(b) or 1(c) is approximately equivalent to the circuit of Fig. 13(c), in which transmission lines approximately $\lambda_o/2$ long are separated by inverters. A lumped-constant equivalent of a line of length θ is shown in Fig. 13(d). This exact equivalent circuit is particularly convenient for θ near 180°. In Fig. 13(e) the equivalent circuit has been simplified by omitting the phase-reversing transformer which has no effect on the insertion-loss response. Also, the series reactances of Fig. 13(f) are small near $\theta = 180°$ and are negligible in comparison with the high characteristic impedances of the inverting elements. Thus, the circuit of Fig. 13(f) is approximately equivalent to that of Fig. 13(c), and hence to the original parallel-coupled-resonator filter.

The analysis will be shortened at this point by comparing the filter circuit of Fig. 13(f) with that shown in Fig. 11(c) of a previous paper.[3] They are seen to be the same, except that the former contains parallel-resonant arms while the latter contains series-resonant arms. Hence, the circuits are duals, and the formulas obtained for the latter may be used in the present analysis if K is replaced by $1/K$, Z_o by $1/Z_o$, and if L and C are interchanged. Thus,

$$\frac{Z_o}{K_{i,i+1}} = \frac{\pi W}{2\omega_1'\sqrt{g_i g_{i+1}}}, \quad i = 1 \text{ to } n-1, \qquad (15)$$

where W is the relative bandwidth defined by (2). For the first and last section of the filter,

$$\frac{Z_o}{K_{01}} = \sqrt{\frac{\pi W}{2\omega_1' g_1}}, \quad \frac{Z_o}{K_{n,n+1}} = \sqrt{\frac{\pi W r}{2\omega_1' g_n}}. \qquad (16)$$

The formulas in Table I follow directly from (13) through (16).